10 Mathematics and the real world

Key words: mass, weight, area, volume, square, cuboid, cube, scale drawing, scale factor, linear dimension, cross-sectional area, surface area, surface area : volume ratio, radius, diameter, circumference, scalar, vector, distance, displacement, speed, velocity, gradient, distance–time graph, displacement–time graph, speed–time graph, velocity–time graph, area under the line (on a graph).

In school mathematics teaching, real-world contexts may be used to help pupils understand abstract ideas as well as how they can be applied. Some of these contexts are the same as those that are also studied in school science. This section looks at such overlaps, in particular those related to the fundamental quantities of mass, length and time.

10.1 Mass and weight

In everyday life, it is quite common to talk about the *weights* of things measured in *grams* (g) or *kilograms* (kg). These are the units shown on familiar items such as kitchen scales or bathroom scales, and it is usual to think of these as devices for weighing things.

In science, however, an important distinction is made between the ***mass*** of an object and the ***weight*** of an object: the *kilogram* is a unit of *mass*, and *weights* are measured in *newtons* (N). It is not that science is correct and everyday language is wrong, but that words are used in different ways in different contexts. Pupils need to understand these differences.

Weight may be the more intuitive concept – heavy objects weigh a lot and are hard to lift up. Weight can be defined scientifically as the gravitational force exerted on an object, and most pupils know that things weigh less on the Moon than on Earth because there is 'less gravity'. However, if an object is taken from the Earth to the Moon, there is still the same amount of 'stuff' or matter in it, even if it weighs less; its mass is a measure of the amount of matter in the object.

Why is this distinction important in science? The following two equations illustrate the difference:

kinetic energy $= \frac{1}{2}mv^2$ (where m = mass and v = velocity)

weight $= mg$ (where m = mass and g = gravitational field strength)

Kinetic energy depends only on an object's mass and velocity. For example, an object of mass 2 kg travelling at 3 m/s has a kinetic energy of 9 J ($\frac{1}{2} \times 2\,\text{kg} \times (3\,\text{m/s})^2$). Its kinetic energy is 9 J whether it is travelling on Earth, on the Moon or in space, since its mass is the same in all of these places.

The weight of an object depends on the gravitational field strength. The value of this is slightly different in different places on the Earth (e.g. in Birmingham it is 9.817 N/kg and in Los Angeles it is 9.796 N/kg), but the average is about 9.81 N/kg. On the Moon, it is much less, at about 1.63 N/kg. A person with a mass of 75 kg would have a weight on Earth of about 736 N (75 kg × 9.81 N/kg) but a smaller weight on the Moon of about 122 N (75 kg × 1.63 N/kg).

Although the weight of this 75 kg person would be slightly different in different places on Earth, the difference is very small (e.g. about 1.002 times heavier in Birmingham than Los Angeles). It is therefore convenient to assume that the gravitational field strength is constant across the Earth, and to treat the weight of an object as being proportional to its mass.

This is the justification for the everyday practice of talking about weights measured in kilograms. It would sound odd and out of place in a shop to talk about 'finding the mass' of some apples rather than 'weighing them'. However, in the school science laboratory, pupils using a balance calibrated in grams should always talk of it as measuring mass.

In school mathematics, it is common to see the term mass used in its scientific sense, but it is possible that pupils may come across books and resources that use the term weight in its everyday sense. In science, it is essential to understand the distinction between mass and weight, as well as being aware of how the terms may be used outside the science classroom.

10.2 Length, area and volume

In mathematics lessons, pupils are likely to have learnt about calculating ***areas*** and ***volumes*** for a variety of two-dimensional and three-dimensional shapes, including the use of units and how to convert from one unit to another. In 11–16 science, pupils also come across calculations of areas and volumes, though for a more limited range of shapes.

For two-dimensional shapes, the following ***formulae*** are used to calculate the areas of a rectangle, a ***square*** (the special case of a rectangle with equal sides), and a right-angled triangle:

area of a rectangle = $a \times b$ (where a and b are the lengths of the sides)

area of a square = a^2 (where a is the length of the side)

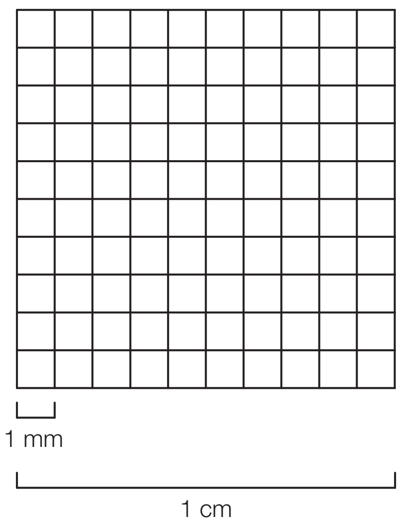
area of a right-angled triangle = $\frac{1}{2}bh$ (where b is the base and h is the height)

When calculating the area of a rectangle, the *units* of the length for each side should be the same. Common units of measurement of length are millimetres (mm), centimetres (cm), metres (m) and kilometres (km). The corresponding units of area are square millimetres (mm^2), square centimetres (cm^2), square metres (m^2) and square kilometres (km^2).

The area of a rectangle of 2 cm by 3 cm is 6 cm^2. What is this area expressed in square millimetres (mm^2)? An easy mistake to make is to think that, since 1 cm = 10 mm, 6 cm^2 = 60 mm^2. Figure 10.1 makes the point that a square with a side of 1 cm contains 100 (10 × 10) squares with a side of 1 mm. Thus, 6 cm^2 = 600 mm^2.

Similarly, in 1 m^2 there are 10 000 cm^2 (100 cm × 100 cm), and in 1 km^2 there are 1 000 000 m^2 (1000 m × 1000 m).

Figure 10.1 An area of 1 cm^2 equals 100 mm^2



For three-dimensional shapes, the following ***formulae*** are used to calculate the volumes of a ***cuboid*** (a shape for which each face is a rectangle) and a ***cube*** (the special case of a cuboid with equal sides):

volume of a cuboid = $a \times b \times c$ (where a, b and c are the lengths of the sides)

volume of a cube = a^3 (where a is the length of the side)

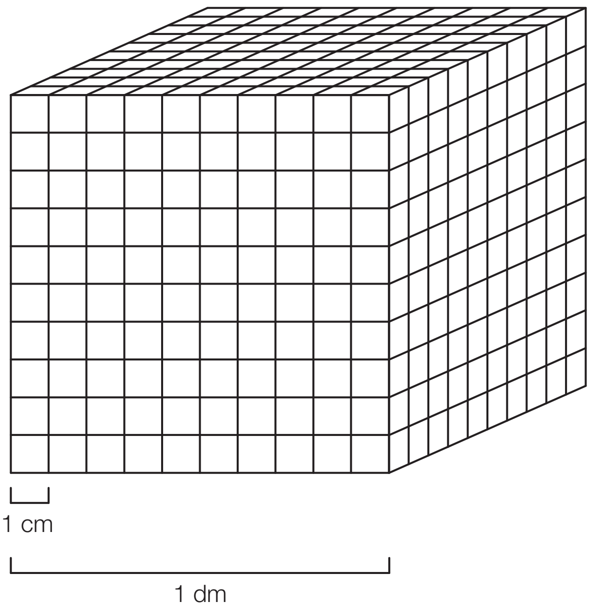
When calculating the volume of a cuboid, the units of the length for each side should be the same. Common units of volume are cubic millimetres (mm^3), cubic centimetres (cm^3), cubic decimetres (dm^3) and cubic metres (m^3).

In everyday life, the volumes of liquids, such as milk or soft drinks, are usually given in millilitres (ml) or litres (l). These units are still encountered in science for liquid measurement, though their use is historical. The accepted units are cubic centimetres ($1\,cm^3 = 1$ ml) and cubic decimetres ($1\,dm^3 = 1$ l). As the name suggests, there are 1000 millilitres in 1 litre; Figure 10.2 illustrates that in $1\,dm^3$ there are $1000\,cm^3$ ($10\,cm \times 10\,cm \times 10\,cm$).

Similarly, in $1\,m^3$ there are $1000\,dm^3$ ($10\,dm \times 10\,dm \times 10\,dm$), and $1\,000\,000\,cm^3$ ($100\,cm \times 100\,cm \times 100\,cm$).

Pupils should understand that the *dimensions* of the unit indicate what quantity is being measured; for example, cm^2 (*two* dimensions) is a measure of *area* while mm^3 (*three* dimensions) is a measure of *volume*.

Figure 10.2 A volume of $1\,dm^3$ equals $1000\,cm^3$

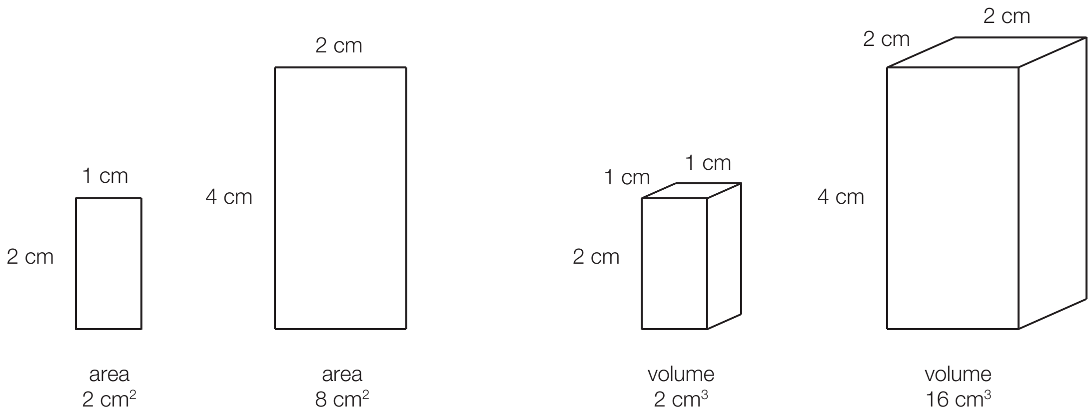


10.3 Scale factor, cross-sectional area and surface area

A ***scale drawing*** is one in which all of the dimensions of the original object are multiplied by a constant ***scale factor***. (Scale factors are discussed in Section 5.9 *Scale drawings and images* on page 48.) For example, in Figure 10.3a, a rectangle with sides of 1 cm and 2 cm has an area of $2\,cm^2$. Re-drawing this with a scale factor of 2 (i.e. doubling the length of each side, called the ***linear dimensions***) gives a rectangle with sides of 2 cm and 4 cm and an area of $8\,cm^2$. Doubling the linear dimensions increases the area not by 2 but by 4 times. (The scale factor is 2, so the area changes by 2^2 times.)

Similarly for a three-dimensional object, doubling the linear dimensions does not result in a simple doubling of the volume. Figure 10.3b shows a cuboid of dimensions $1\,cm \times 1\,cm \times 2\,cm$, giving a volume of $2\,cm^3$. Doubling the linear dimensions of the object gives a volume of $16\,cm^3$ ($2\,cm \times 2\,cm \times 4\,cm$). Thus, doubling the linear dimensions increases the volume not by 2 but by 8 times. (The scale factor is 2, so the volume changes by 2^3 times.)

Figure 10.3 Effects of scaling



In summary, when an object is scaled:

change in the linear dimensions $\propto$ scale factor

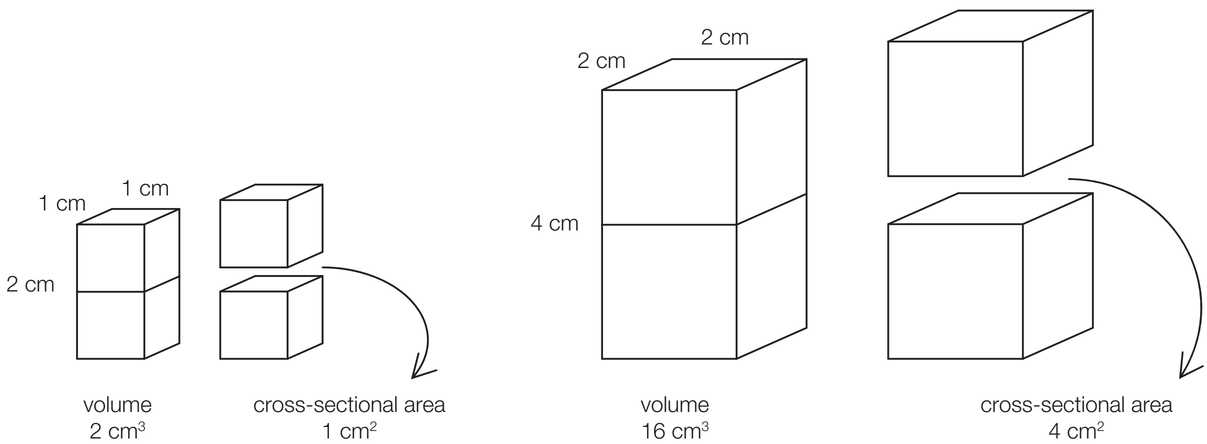
change in the area $\propto$ (scale factor)^2

change in the volume $\propto$ (scale factor)^3

The examples given in Figure 10.3 relate to a *doubling* of the linear dimensions (a scale factor of 2) but the same principles apply to other scale factors. For example, if the linear dimensions are *trebled* (a scale factor of 3) then the area increases by 9 times (3^2) and the volume increases by 27 times (3^3).

These scaling effects are important in many areas of science, particularly biology. For example, it explains why the legs of an elephant are much thicker relative to its body size than those of a mouse. Figure 10.4 shows how doubling the linear dimensions of a cuboid affects its volume and its **cross-sectional area**. The volume increases *8 times*, but the cross-sectional area only *4 times*.

Figure 10.4 Cross-sectional area and scaling



The strength of an animal's legs is related to their cross-sectional area, while the weight of the animal is related to its volume. If a mouse were to be scaled up in size, its legs would not be strong enough to support its weight. It is because weight increases faster than strength that an elephant's legs are relatively much thicker.

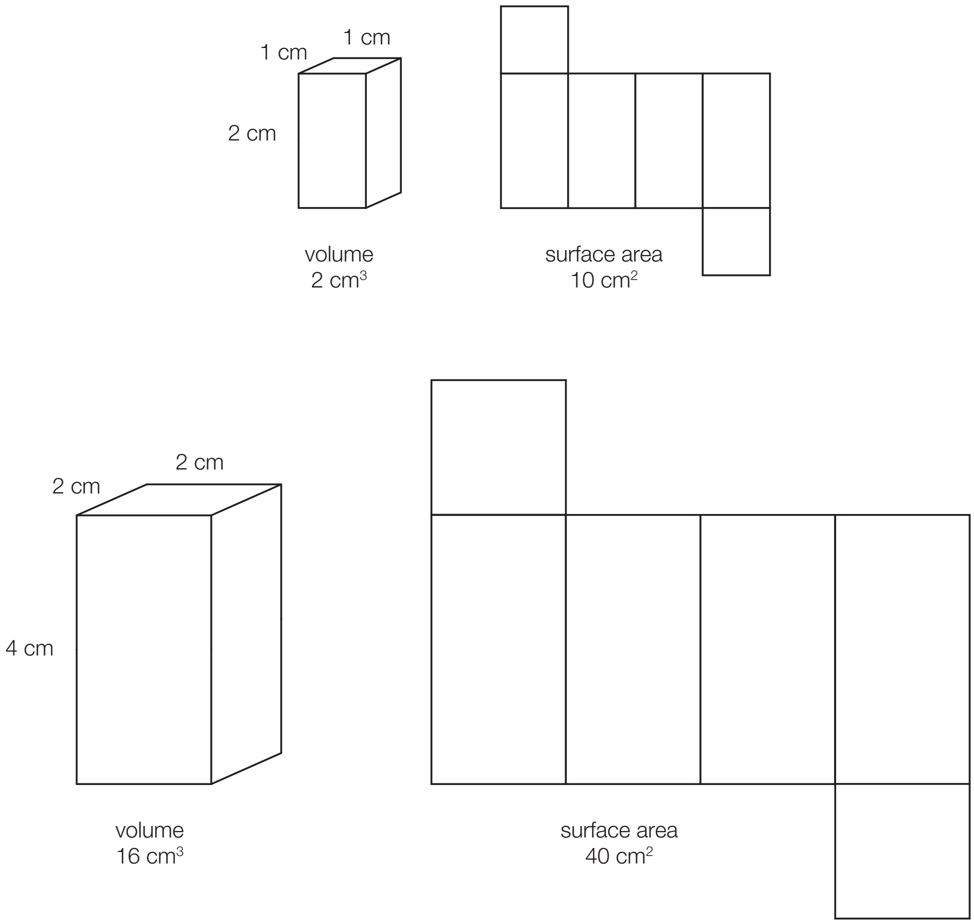
Another similar example is the limit placed on the size of a biological cell. The surface area of a cell must be sufficient for substances to diffuse into and out of the cell fast enough.

Figure 10.5 shows the effect of doubling the size of a cuboid on its volume and on its ***surface area***. As in the previous example, the volume increases *8 times*, but the surface area only *4 times*.

This idea is usefully expressed in terms of the ***surface area : volume ratio***. Since the change in the area is proportional to (scale factor)^2 and the change in the volume is proportional to (scale factor)^3, this means that the change in the surface area : volume ratio is inversely proportional to the scale factor. That is, *doubling* the linear dimensions leads to a *halving* of the surface area : volume ratio.

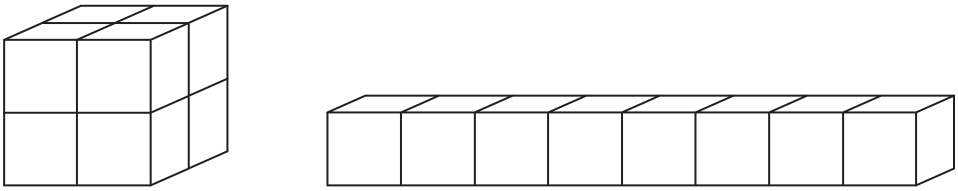
If a biological cell is scaled up in size, its surface area : volume ratio gets smaller, and it is this that puts a limit on the size of a cell. Substances are not able to move in and out through the surface of the cell fast enough for its volume.

Figure 10.5 Effect of size on surface area : volume ratio



The surface area : volume ratio is also affected by the *shape* of an object. If you have eight cubes each of 1 cm^3, there are various ways of arranging them. Whichever way they are arranged, they always have the same total volume (8 cm^3) but the surface areas may be different. To have the smallest surface area, they need to be arranged in a cube ($2 \times 2 \times 2$), as shown on the left of Figure 10.6. Counting the number of squares on each face shows that this has a total surface area of 24 cm^2. The arrangement with the largest surface area ($8 \times 1 \times 1$) is shown on the right. This has a surface area of 32 cm^2.

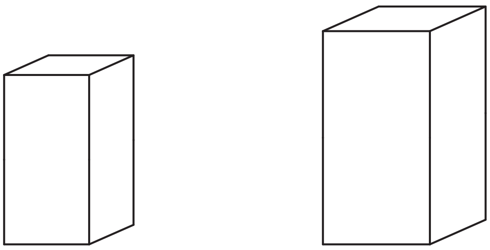
Figure 10.6 Effect of shape on surface area : volume ratio



An example of this in the real world is how to keep warm in cold conditions. It is better to try to roll up into a ball, thus reducing your surface area from which heat can escape. (Note that a sphere has a smaller surface area than a cube of the same volume. A sphere is the shape that has the smallest possible surface area : volume ratio.)

Human perception is not good at comparing the volumes of objects. The drawing in Figure 10.7 represents two objects, the second of which is twice the volume of the first. It is not easy to judge this by eye. Talking about 'doubling the size of an object' is ambiguous if it is not made clear whether this is referring to the linear dimensions, the area or the volume.

Figure 10.7 It is difficult to compare the volumes of objects



On a bar chart it is relatively easy to compare the sizes of the bars because we only need to pay attention to the *length* of the bars. Some 'informal' graphical displays replace the bars with different sized 3D representations of an object that are related to the quantity being plotted (pictograms). For example, electricity consumption may be represented by different sized light bulbs. Because of the difficulties in making the comparisons, such displays can be misleading (and indeed may sometimes be used to mislead deliberately). Using 3D representations in bar charts is best avoided.

10.4 Circles and spheres

Modelling biological aspects of the world using squares and cubes may be convenient, but in nature such shapes are less common than circles and spheres. However, calculations involving these (which always involve π) are not so easy to handle and are not much used in 11–16 science, though pupils should be familiar with the formulae for doing such calculations from their mathematics lessons.

Mathematically, a circle and a sphere are defined in terms of the set of points that are a fixed distance (the ***radius***) from the centre. However, for a real object, such as a coin or a ball bearing, it is the ***diameter*** that is more easily measured. So, while in mathematics the formulae used are usually based on the radius, in science the context determines whether it is more useful to use radius or diameter.

In the following formulae, the letter r represents the radius:

$$\text{diameter of a circle} = 2r$$

$$\text{circumference of a circle} = 2\pi r$$

$$\text{area of a circle} = \pi r^2$$

$$\text{surface area of a sphere} = 4\pi r^2$$

$$\text{volume of a sphere} = \tfrac{4}{3}\pi r^3$$

An important point about such formulae is that the power to which r is raised (r, r^2 or r^3) is a clue to the nature of what is being calculated:

- the ***diameter*** and ***circumference*** of a circle are ***linear dimensions*** and are proportional to r
- the *area* of a circle and the *surface area* of a sphere are proportional to r^2
- the *volume* of a sphere is proportional to r^3.

These relationships mean that scaling effects are the same for a sphere as for a cube so, in terms of modelling, a cube is just as good a shape as a sphere. In fact, since cubes can be stacked together into different shapes in a way that spheres cannot, they are more useful in modelling scaling effects.

10.5 Scalars and vectors: distance and displacement

Some quantities have both a *size* and a *direction*. A force is an example – its size can be measured in newtons (N), and it also acts in a particular direction. It is called a ***vector*** quantity. Other quantities, such as volume, have a size but no direction and are called ***scalar*** quantities.

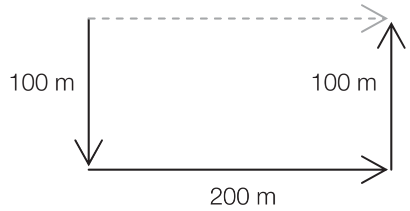
This distinction, between vector and scalar quantities, arises when thinking about the movement of things from one place to another. For example, imagine you walk the path as illustrated in Figure 10.8.

Figure 10.8 A simple path

A: 100 metres South

B: 200 metres East

C: 100 metres North



There are two ways of thinking about *how far you have gone*. The first is to think about the length of the path you have walked – in total, 400 metres (100 metres + 200 metres + 100 metres). The second is to think about how far you have ended up from where you started. This is shown by the dotted line: 200 metres East of the start.

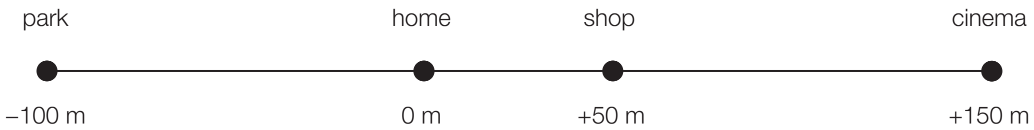
The scientific terms for these two ways of expressing how far you have gone are ***distance*** and ***displacement***:

- *Distance*: This is a scalar quantity. It has a size but no direction. For example, the distance for part A of the journey is 100 metres. The total distance for the whole journey (400 metres) can be found by adding the values of the distances for each part of the journey together.
- *Displacement*: This is a vector quantity. It has both a size and a direction. For example, the displacement for part A of the journey is 100 metres South. Finding the displacement for the whole journey (200 metres East) involves more than just adding the sizes together, since the direction needs to be taken into account.

Adding displacements together gets even more complicated if they can be at any angle to each other, not just right angles. This involves using trigonometry (sines, cosines, and so on). This kind of addition of vectors is important in post-16 physics but, for 11–16 science, *calculations* on vector addition are made simple by *only working in one dimension*. However, pupils at this level may be expected to know how to represent the addition of vectors *graphically*, by making scale drawings of situations involving forces.

Figure 10.9 shows an example of vectors in one dimension. It shows displacements for various locations relative to a person's home (shown as 0 m). In this diagram, displacements to the right are indicated by a 'plus' sign, and those to the left by a 'minus' sign (rather than using terms like East and West to indicate direction). This is a very commonly used convention.

Figure 10.9 Displacements in one dimension



Thus, travelling from home to the shop is a displacement of +50 m, and travelling from the shop to the cinema is a *further* displacement of +100 m. To go from the shop to the park is a total *distance* of 150 m, and as this is in the *left* direction, the *displacement* is −150 m (i.e. negative). Going in the opposite direction, from the park to the shop, is the same distance (150 m), but the displacement is +150 m (i.e. positive). In order to be able to manipulate such vectors, pupils need to know how to add and subtract positive and negative numbers (see Section 9.3 *Operations and symbols* on page 90).

Note that, although working in one dimension makes things simpler, it also means that the vector/scalar distinction is more subtle. The only difference between a distance and a displacement is whether or not there is a sign (+ or −) in front of the value. For movement in two or three dimensions, the differences are more obvious, as the direction is stated in full. However, using the terms distance and displacement correctly is essential. If not, it leads to confusion when it comes to doing calculations and drawing graphs. Unfortunately, this distinction is not always made sufficiently clear.

10.6 Movement of objects: speed and velocity

The ***speed*** of a moving object is defined as the *distance* it travels in unit time, and the formula is:

$$\text{speed} = \frac{\text{distance}}{\text{time}}$$

Since distance is a ***scalar*** quantity (i.e. it does not have a direction), speed is also a scalar quantity. The term for *speed in a particular direction* is ***velocity*** – this is a ***vector*** quantity and is found from this formula:

$$\text{velocity} = \frac{\text{displacement}}{\text{time}}$$

As its name suggests, the speedometer on a car measures *speed*. A car going at a constant speed along a straight motorway is also moving at a constant velocity, since *its direction stays* *the same*. However, if it goes round a corner at a constant speed (so that the reading on the speedometer stays the same), its velocity is *not* constant. The velocity is continually changing around a corner since *its direction is changing*.

The formulae for speed and velocity can be rearranged to give the following two equations:

$$\text{distance} = \text{speed} \times \text{time}$$

$$\text{displacement} = \text{velocity} \times \text{time}$$

The equations both have the form $y = mx$. The first of these equations shows that, if we plot a line graph of *distance* against *time* for a moving object, the **gradient** of the line is its *speed*. Similarly, the second equation shows that if we plot a line graph of *displacement* against *time* for an object moving in one dimension, the gradient of the line is its *velocity*. Such graphs are very useful for showing the behaviour of a moving object, and will be illustrated with an example.

Figure 10.10 shows the journey of a cyclist who travels from home to the cinema, then to the park and back home. (Since the three places are in a straight line it is a one-dimensional journey.) Note that the total distance that the cyclist travels is 500 m (150 m + 250 m + 100 m); however, the total displacement is *zero*, because the cyclist ends up in the same place as at the start.

Figure 10.10 A one-dimensional journey

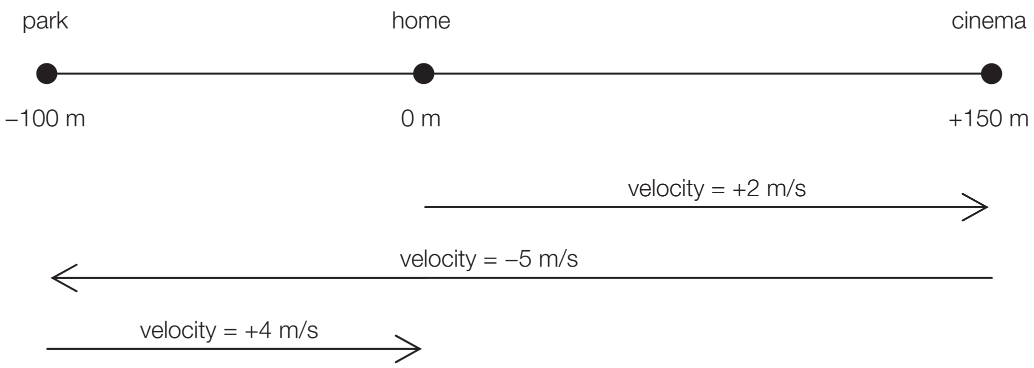


Figure 10.11a shows a **distance–time graph** for this journey. The gradients for each line segment indicate the *speeds* for each part of the journey. The cyclist starts slowly, then speeds up for the second part, and slows down a little at the end. Note that on a distance–time graph, the value for the distance must always get larger over time (you cannot 'undo' the distance travelled), so the gradient of the line is never negative (i.e. it never slopes downwards).

Figure 10.11b shows a **displacement–time graph** for the journey. Although the journey is the same, the appearance of the graph is very different. Here, the gradients for each line segment indicate the *velocities* for each part of the journey. Initially, the velocity is positive (the gradient is positive and slopes upwards), but then the velocity becomes negative (the gradient is negative and slopes downwards). In other words, the cyclist changes direction. After another change in direction, the velocity is positive again and the final value of the displacement is zero (the cyclist is home).

Although the term 'displacement–time graph' is common in school science, strictly speaking, such a graph cannot be drawn since displacement is a *vector*. A graph cannot show both the *size* and *direction* of a quantity. What is called a 'displacement–time graph' actually shows how the *size of the component in a chosen direction of the displacement* of an object changes over time. Such graphs are useful only for objects *moving in a straight line*. A similar point also applies to what are called 'velocity–time graphs' (discussed below), since velocity is also a vector.

In mathematics, the displacement here might be referred to as the 'distance of the cyclist from home', but for a one-dimensional journey these are essentially the same. At the end of the journey, the cyclist's distance from home is zero, and the graph would still have the same shape.

In summary, plotting distance and displacement on a graph can show how these quantities change over time. The rate of change of distance is speed, and the rate of change of displacement is velocity. It is also possible to plot *speed* and *velocity* on a graph to see how these change over time as well.

Figure 10.11 The same journey represented in different ways

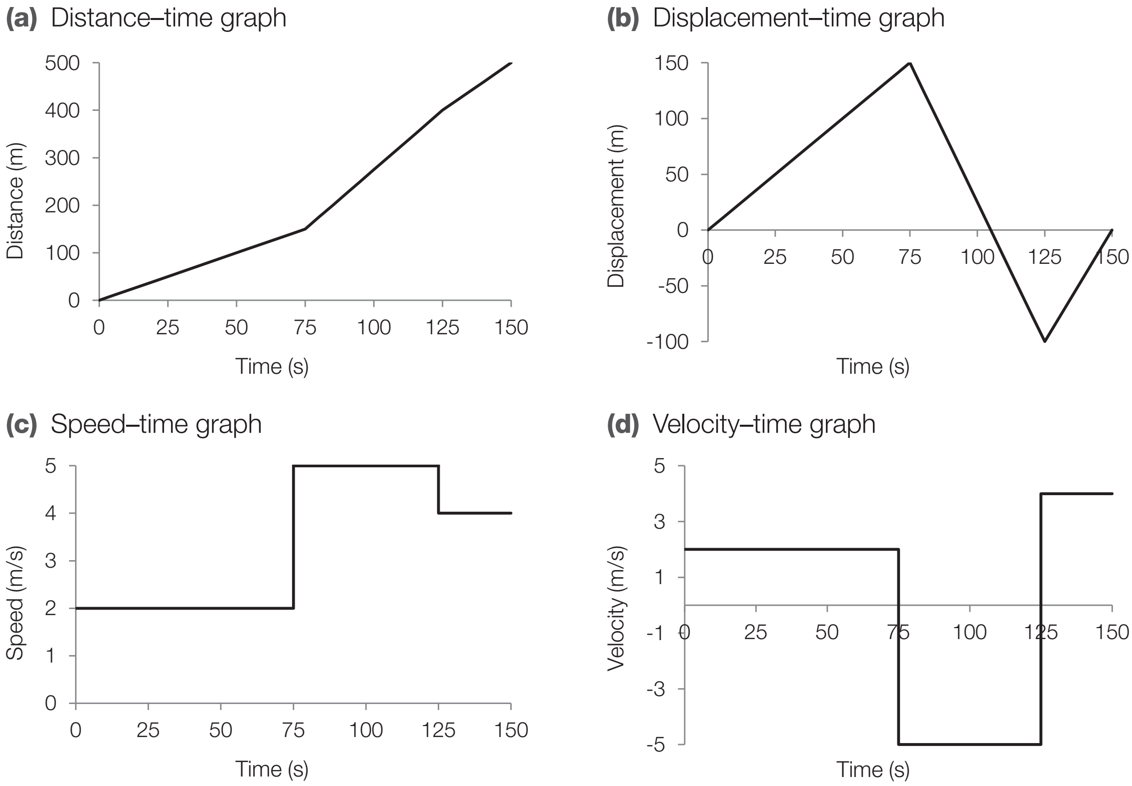


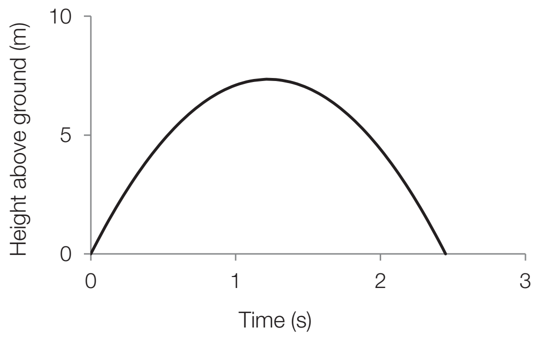
Figure 10.11c shows a ***speed–time graph***, again for the same journey. It shows that, for each of the three stages, the cyclist was travelling at different constant speeds, i.e. each of the lines is horizontal. Figure 10.11d shows a ***velocity–time graph***. This also shows three horizontal lines representing constant velocities for each of these stages. The difference here is that the second of these lines is *below* the horizontal axis, indicating that the velocity is *negative*.

10.7 Gradients of lines on speed–time and velocity–time graphs

The gradient of a line on a speed–time graph or a velocity–time graph indicates the rate at which the speed or velocity is changing. This is called *acceleration*. The graphs shown in Figure 10.11 are idealised and do not represent what a real journey would look like, since the changes in speed or velocity happen in zero time. The lines on the graph are vertical: this implies that the acceleration is infinitely large.

A more realistic situation to illustrate the meaning of acceleration is a ball being thrown vertically upwards from the ground and then falling back down to the ground. The change in the ball's height with time is show in Figure 10.12. The changing gradient tells us that, as the ball gets higher, it gets slower and slower until it reaches its maximum height; it then gets faster and faster until it reaches the ground.

Figure 10.12 A ball thrown vertically upwards



Strictly speaking, *acceleration* is the rate of change of *velocity* (i.e. it is a *vector* quantity), and can be calculated using the formula:

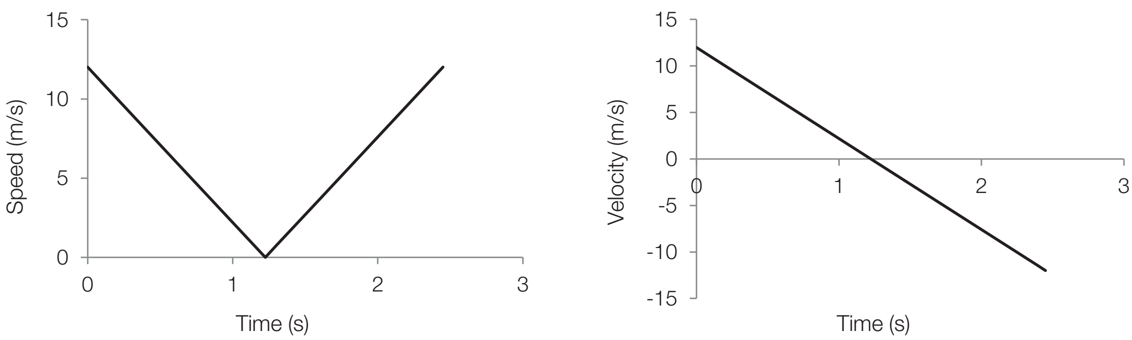
$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time}}$$

However, in school science, it is also commonly used to mean the rate of change of *speed* (i.e. a *scalar* quantity). Because the same word is used to mean two different things, it is important that the context makes it clear whether it is referring to the rate of change of speed or of velocity. A helpful way of making this difference explicit is to talk of a *scalar acceleration* (rate of change of speed) or a *vector acceleration* (rate of change of velocity).

Figure 10.13 shows a speed–time graph and a velocity–time graph for the ball thrown upwards.

Figure 10.13 Representing speed and velocity for a ball thrown vertically upwards

(a) Speed–time graph **(b)** Velocity–time graph



The speed–time graph in Figure 10.13a shows that the ball's speed constantly decreases until it reaches zero (the maximum height) and then steadily increases; in other words, it decelerates and then accelerates. The gradient of the graph represents a *scalar acceleration*: for the first part it is *negative* (the ball is slowing down, or decelerating) and for the second part it is *positive* (the ball is getting faster, or accelerating).

Note that, while it is useful to talk about an object *accelerating* or *decelerating*, the term *deceleration* is better avoided, since only the term *acceleration* represents a quantity with a value.

The velocity–time graph in Figure 10.13b uses the convention that positive values of velocity mean 'going up' and negative values mean 'going down' (values of displacement are taken as positive above ground and negative below it). Here, the gradient has the same *negative* value throughout – it represents a constant *vector acceleration* in the *downwards* direction. The velocity starts with a positive value, decreases until it becomes zero, and continues to decrease when it becomes negative.

Thus, the meaning of a positive or negative acceleration depends on the way the term is being used.

- *Scalar acceleration*: A positive acceleration means getting faster; a negative acceleration means getting slower.
- *Vector acceleration*: The sign (i.e. direction) of the acceleration *on its own* gives no indication whether the ball is getting faster or slower – it depends on the direction of the velocity. An acceleration in the same direction as the velocity (both positive or both negative) means getting faster; an acceleration in a different direction to the velocity (one is positive, the other negative) means getting slower.

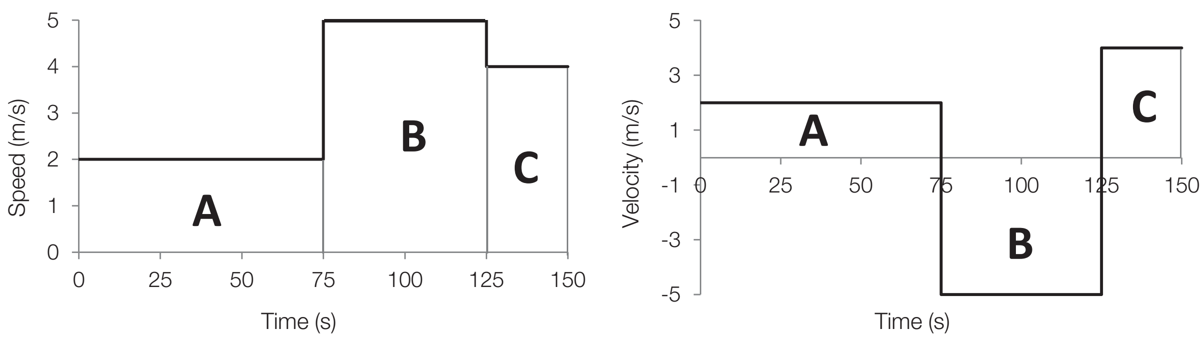
10.8 Area under the line on speed–time and velocity–time graphs

On a graph showing a rate of change against time, the ***area under the line*** is meaningful (see Section 9.13 *Graphs of rates against time: area under the line* on page 105). For a ***speed–time graph***, the vertical axis represents the *rate of change of distance* (speed) and the horizontal axis represents *time*. The area under the line then represents *distance*. Figure 10.14a shows the speed–time graph with the areas for each stage of the journey marked. The area of 'A' is 2 m/s × 75 s. This gives 150 m – the distance travelled in this stage of the journey. Calculating the areas of 'B' and 'C' and then adding all the areas together will give the total distance travelled.

In a similar way, displacement can be found by adding together the areas on a ***velocity–time graph***. However, in this case, since velocity can have both positive and negative values, so too can the areas. In Figure 10.14b, the areas of 'A' and 'C' are positive, but 'B' is negative (it is below the horizontal axis). Since for this journey the displacement is zero, the two areas on this graph above the line are equal to the area below, and when they are all added together the total area is zero.

Figure 10.14 Using areas to find distance or displacement

(a) Distance from a speed–time graph **(b)** Displacement from a velocity–time graph



The area under the line on a distance–time graph (distance × time) does not have any real-world meaning, and the same applies to the area under the line on a displacement–time graph (displacement × time). Thus, while the *gradients* of the lines on the four graphs shown in Figure 10.11 all have a real-world meaning, the *areas under the line* are meaningful only for the speed–time and velocity–time graphs.